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Strong Edge-Graceful Labeling of Disconnected Graphs

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Abstract-For a graph to be edge graceful there is a condition which can be found by Lo .Strong edge graceful labeling was introduced by gayathri et.al. Here we discussed strong edge graceful of disconnected graphs.

Keywords- Strong edge graceful labeling; graceful labeling; disconnected graphs
AMS Subject Classification— 05C78.

1. Introduction

By a (p, q) graph G, we mean a graph G = (V, E) with | V | = p and | E | = q. There is a survey in The Electronic journal of Combinatorics for graph labelling . Which contains various papers which consists of applications and various types of labeling which is updated every year. Lo[4] derived a necessary condition for a graph to be edge graceful. In [1] Gayathri define a labeling called strong edge graceful labeling.

A (p,q) graph G is said to have a **strong edge-graceful labeling (SEGL)** if there exists an injection f from the edge set to the set $\{1,2,\dots,\left[\frac{3q}{2}\right]\}$ so that the induced mapping f^+ from the vertex set to $\{0,1,2,\dots,2p-1\}$ defined by a $f^+(x)=\sum\{f(xy)|xy\in E(G)\}(mod\ 2p)$ are distinct. A graph G is said to be a **strong edge-graceful graph (SEGG)** if it admits a strong edge-graceful labeling. Here , $[x]$ denotes the integer part of x.

In[2]Strong edge graceful labeling in the context of a switching of a vertex and, In [3] strong edge–graceful labeling of shadow and splitting graph has been studied. In this paper we discuss strong edge–graceful labeling of strong edge-graceful labeling of disconnected graphs

2. Strong edge-graceful labeling.

Theorem 2.1

The graph $P_m^+\cup P_n, (m\,,n\geq 3)$ is a strong edge-graceful graph.

Proof

Let $\{v_i,v_i',u_j,|1\leq i\leq m, 1\leq j\leq n\}$ be the vertices and $\{e_i,e_j'|\ 1\leq i\leq 2m-1, 1\leq j\leq n-1\}$ be the edges of $P_m^+\cup P_n$ as shown in Figure 2.1.

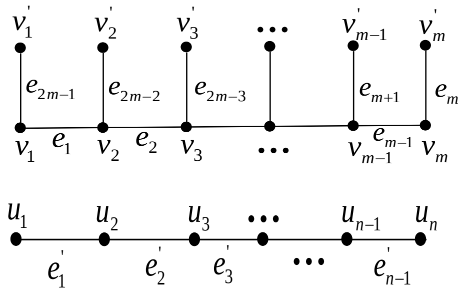

Figure 2.1: Ordinary labeling of $P_m^+\cup P_n$

Case 1: $m\leq n$ (except for $P_3^+\cup P_6$)

We first label the edges as follows:

Define $f: E(P_m^+\cup P_n)\to\{1,2,\dots,\left[\frac{3q}{2}\right]\}$ by

$f(e_i)=i, \qquad 1\leq i\leq 2m-1,\ i\neq m,\ m-1$

$f(e_m)=m-1;\ f(e_{m-1})=2m+1$

$f(e_1')=m;$

$f(e_i')=i+2m,\ 2\leq i\leq n-2\ \ (n\geq 4)$

For $m\,,n$ even (or) m odd and n odd, $n\neq 2m+1$ (or) m odd, $n=2m$ (or) m even, n odd, $n\neq m+1, 2\,m+1, m+3$

$f(e_{n-1}')=2m+n-1$

For m odd, n even and $n\neq 2m$ (or) m odd, n even $n=2m+1$ (or) m even, n odd and $n=m+1, m+3, 2m+1$

$f(e_{n-1}')=2m+n$

Then the induced vertex labels are:

$f^+(v_i)=i+2m-1, \qquad 1\leq i\leq m-2$

$f^+(v_{m-1})=4m\,; \qquad f^+(v_m)=3m$

$f^+(v_i')=2m-i, \qquad 1\leq i\leq m-1$

$f^+(v_m')=m-1; \qquad f^+(u_1)=m$

$f^+(u_2)=3m+2$

$f^+(u_i)=2i+4m-1, \quad 3\leq i\leq n-2\ (n\geq 5)$

For $m\,,n$ even (or) m odd and n odd, $n\neq 2m+1$ (or) m odd, $n=2m$ (or) m even, n odd, $n\neq m+1, 2\,m+1, m+3$

$f^+(u_{n-1})=4m+2n-3$

$f^+(u_n) = 2m + n - 1$

For m odd, n even, and $n \neq 2m$ (or) m odd, n even $n = 2m+1$ (or) m even, n odd and $n = m+1, m+3, 2m+1$

$f^+(u_{n-1}) = 4m + 2n - 2$

$f^+(u_n) = 2m + n$

Case 2: $n < m$

$f(e'_i) = i, \qquad 1 \le i \le n-2$

$f(e'_{n-1}) = \begin{cases} n, & n \text{ even} \\ n-1 & n \text{ odd} \end{cases}$

$f(e_i) = i + n, \qquad 1 \le i \le 2m-1$

Then the induced vertex labels are:

$f^+(u_i) = 2i - 1, \qquad 1 \le i \le n-2$

$f^+(u_{n-1}) = \begin{cases} 2n-2, & n \text{ even} \\ 2n-3, & n \text{ odd} \end{cases}$

$f^+(u_n) = \begin{cases} n, & n \text{ even} \\ n-1, & n \text{ odd} \end{cases}$

$f^+(v_1) = 2m + 2n$

$f^+(v_i) = 2m + 3n + i - 1, 2 \le i \le m-1$

$f^+(v_m) = 2m + 2n - 1$

$f^+(v'_i) = 2m + n - i, \qquad 1 \le i \le m$

Case 3: $m = 3, n = 6$

Strong edge-graceful labeling of $P_3^+ \cup P_6$ is shown in Figure 2.2.

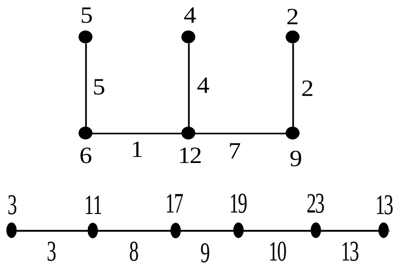

Figure 2.2: SEGL of $P_3^+ \cup P_6$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^+ : V(P_m^+ \cup P_n) \to \{0,1,2,\ldots,2p-1\}$. Hence, f is a strong edge-graceful labeling.

Thus, the graph $P_m^+ \cup P_n$ is a strong edge-graceful graph for all $m, n \ge 3$.

Illustration 2.2

Strong edge-graceful labeling of $P_7^+ \cup P_9$,

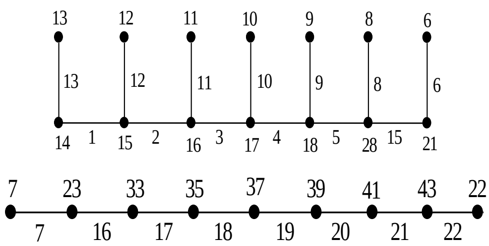

Figure 2.3: SEGL of $P_7^+ \cup P_9$

Theorem 2.3

The graph $P_m^+ \cup C_n$, $(m, n \ge 3)$ is a strong edge-graceful graph.

Proof

Let $\{u_i, v_j, v'_j \mid 1 \le i \le n, 1 \le j \le m\}$ be the vertices and $\{e'_i, e_j \mid 1 \le i \le n, 1 \le j \le 2m-1\}$ be the edges of $P_m^+ \cup C_n$ as shown in Figure 2.4.

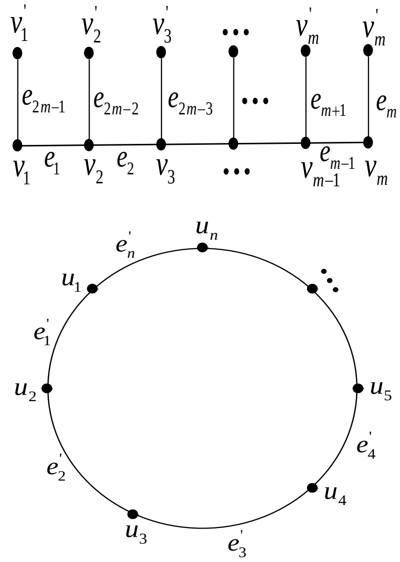

Figure 2.4: Ordinary labeling of $P_m^+ \cup C_n$

Case 1: $m \le n$ $(m \ge 3, n \ge 4)$

We first label the edges as follows:

Define $f: E(P_m^+ \cup C_n) \to \{1,2,\ldots,\left\lceil \frac{3q}{2} \right\rceil\}$ by

$f(e_i) = i, \qquad 1 \le i \le 2m-1,\ i \neq m,\ m-1$

$f(e_m) = m - 1;\ f(e_{m-1}) = 2m + 1$

$f(e'_1) = m;$

$f(e'_i) = i + 2m,\ 2 \le i \le n-2$

For m, n even, $m \neq n$ (or) m, n odd, $m \neq n$ (or) m even n odd

$f(e'_{n-1}) = 2m + n - 1$

For m odd, n even (or) $m = n$

$f(e'_{n-1}) = 2m + n$

$f(e'_n) = 2m + n + 1$

For m, n even, $m \neq n$ (or) m, n odd, $m \neq n$

$f(e'_n) = 2m + n$

For m even and n odd

$f(e'_n) = 2m + n + 1$

Then the induced vertex labels are:

$f^+(v_i) = i + 2m - 1, \qquad 1 \le i \le m-2$

$f^+(v_{m-1}) = 4m; \qquad f^+(v_m) = 3m;$

$f^+(v'_i) = 2m - i, \qquad 1 \le i \le m-1$

$f^+(v'_m) = m - 1; \qquad f^+(u_2) = 3m + 2$

$f^+(u_i) = 4m + 2i - 1, \qquad 3 \le i \le n-2$

For m, n even (or) m, n odd and $m \neq n$

$f^+(u_1) = 3m + n;$

$f^+(u_{n-1}) = 4m + 2n - 3$

$f^+(u_n) = 4m + 2n - 1$

For m even, n odd (or) $m = n$

$f^+(u_1) = 3m + n + 1;$

$f^+(u_{n-1}) = 4m + 2n - 3$

$f^+(u_n) = 0$

For m odd, n even

$f^+(u_1) = 3m + n + 1;$

$f^+(u_{n-1}) = 4m + 2n - 2$

$f^+(u_n) = 1$

Case 2: $n < m$ $(n \geq 3, m \geq 4)$

$f(e_i') = i, \qquad 1 \leq i \leq n-2$

$f(e_{n-1}') = \begin{cases} n, & n \text{ } even \\ n-1, & n \text{ } odd \end{cases}$

$f(e_n') = \begin{cases} 2m+2n, & n \text{ } odd \\ n-1 & n \text{ } even \end{cases}$

$f(e_i) = i + n, \qquad 1 \leq i \leq 2m-1$

Then the induced vertex labels are:

$f^+(v_1) = 2m + 2n$

$f^+(v_i) = 2m + 3n + i - 1, 2 \leq i \leq m-1$

$f^+(v_m) = 2m + 2n - 1$

$f^+(v_i') = 2m + n - i, \qquad 1 \leq i \leq m$

$f^+(u_1) = \begin{cases} n, & n \text{ } even \\ 2m+2n+1, & n \text{ } odd \end{cases}$

$f^+(u_i) = 2i - 1, \qquad 2 \leq i \leq n-2$

$f^+(u_{n-1}) = \begin{cases} 2n-2, & n \text{ } even \\ 2n-3, & n \text{ } odd \end{cases}$

$f^+(u_n) = \begin{cases} 2n-1, & n \text{ } even \\ 2m+3n-1, & n \text{ } odd \end{cases}$

Case 3: $m = n = 3$

Strong edge-graceful labeling of $P_3^+ \cup C_3$ is shown in Figure 2.5.

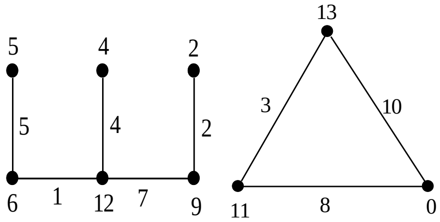

Figure 2.5: SEGL of $P_3^+ \cup C_3$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function

$f^+: V(P_m^+ \cup C_n) \to \{0,1,2,\dots,2p-1\}$. Hence, f is a strong edge-graceful labeling.

Thus, the graph $P_m^+ \cup C_n$ is a strong edge-graceful graph for all $m, n \geq 3$.

Illustration 2.4

Strong edge-graceful labeling of $P_6^+ \cup C_6$,

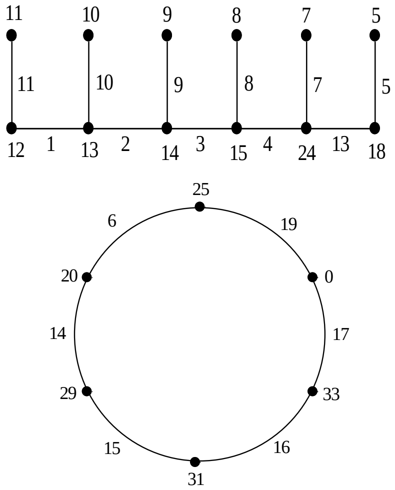

Figure 2.6: SEGL of $P_6^+ \cup C_6$

Theorem 2.5

The graph $C_n^+ \cup C_m$, $(n, m \geq 3)$ is a strong edge-graceful graph.

Proof

Let $\{u_i, u_i', v_j | 1 \leq i \leq n, 1 \leq j \leq m\}$ be the vertices and $\{e_i', e_j \mid 1 \leq i \leq m, 1 \leq j \leq 2n\}$ be the edges of $C_n^+ \cup C_m$ as shown in Figure 2.7. We note that $|V(C_n^+ \cup C_m)| = 2n + m$ and $|E(C_n^+ \cup C_m)| = 2n + m$.

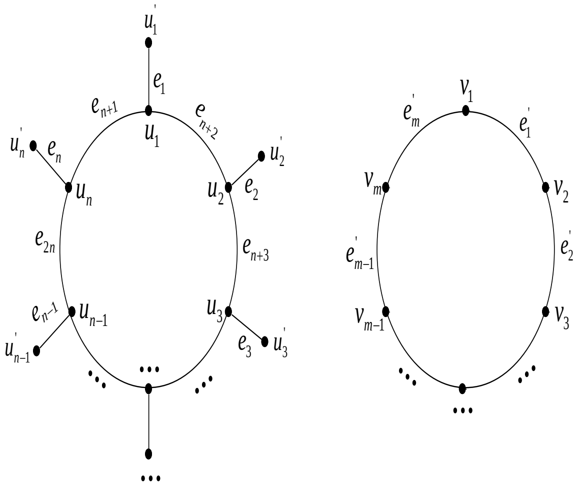

Figure 2.7: Ordinary labeling of $C_n^+ \cup C_m$

We first label the edges of $C_n^+ \cup C_m$ as follows:

Define $f: E(C_n^+ \cup C_m) \to \{1,2,\dots,\left\lceil \frac{3q}{2} \right\rceil\}$ by

$f(e_i) = i + 1, \qquad 1 \leq i \leq n$

$f(e_{n+1}) = 1$

$f(e_i) = 3n - i + 2, \qquad n+2 \leq i \leq 2n$

$f(e_i') = i + 2n, \qquad 1 \leq i \leq m-1$

$f(e_m') = \begin{cases} m+2n, & m \text{ } odd \\ m+2n+1, & m \text{ } even \end{cases}$

Then the induced vertex labels are:

$f^+(u_i') = i + 1, \qquad 1 \leq i \leq n$

$f^+(u_1) = 2n + 3$

$f^+(u_i) = 4n - i + 4, \qquad 2 \le i \le n-1$
$f^+(u_n) = 2n + 4$
$f^+(v_1) = \begin{cases} 4n + m + 1, & m\ odd \\ 4n + m + 2, & m\ even \end{cases}$
$f^+(v_i) = 4n + 2i - 1, \qquad 2 \le i \le m-1$
$f^+(v_m) = \begin{cases} 4n + 2m - 1, & m\ odd \\ 0, & m\ even \end{cases}$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^+ : V(C_n^+ \cup C_m) \to \{0,1,2,\dots,2p-1\}$. Hence, f is a strong edge-graceful labeling.

Thus, the graph $C_n^+ \cup C_m$ is a strong edge-graceful graph for all $n, m \ge 3$.

Illustration 2.6

Strong edge-graceful labeling of $C_6^+ \cup C_7$

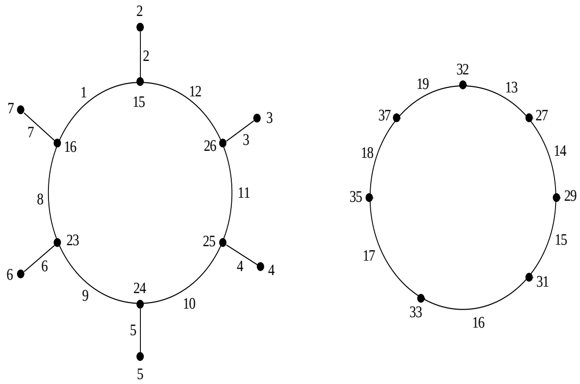

Figure 2.8: SEGL of $C_6^+ \cup C_7$

Theorem 2.7

The graph $P_m^+ \cup P_n^+$, $(m \ge 4, n \ge 3)$ is a strong edge-graceful graph.

Proof

Let $\{u_i, u_i', v_j, v_j' \mid 1 \le i \le m, 1 \le j \le n\}$ be the vertices and $\{e_i, e_j' \mid 1 \le i \le 2m-1, 1 \le j \le 2n-1\}$ be the edges of $P_m^+ \cup P_n^+$ as shown in Figure 2.9.

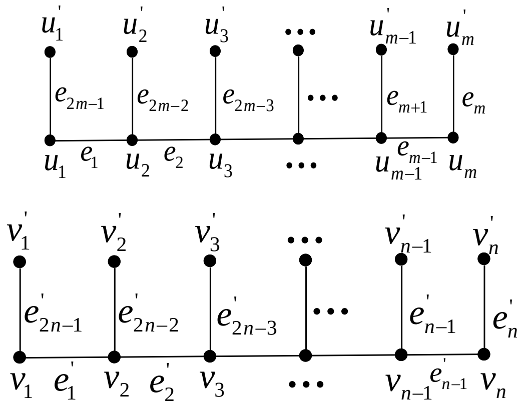

Figure 2.9: Ordinary labeling of $P_m^+ \cup P_n^+$

Without loss of generality, let $m \ge n$.
We first label the edges as follows:

Define $f : E(P_m^+ \cup P_n^+) \to \left\{1,2,\dots,\left[\frac{3q}{2}\right]\right\}$ by

$f(e_i) = i, \qquad 1 \le i \le m-1$
$f(e_m) = 3(m+n) - 4$
$f(e_i) = 2n + i - 1, \qquad m+1 \le i \le 2m-1$
$f(e_1') = 2n + 2m - 1$
$f(e_i') = i - 1 + m, \qquad 2 \le i \le 2n-2, \quad i \ne n$
$f(e_n') = 3(m+n) - 3$;
$f(e_{2n-1}') = m + n - 1$

Then the induced vertex labels are:

$f^+(u_i) = i + 2n + 2m - 2, \qquad 1 \le i \le m-1$
$f^+(u_m) = 4m + 3n - 5$
$f^+(u_i') = 2n + 2m - i - 1, \qquad 1 \le i \le m-1$
$f^+(u_m') = 3(m+n) - 4$;
$f^+(v_1) = 3m + 3n - 2$
$f^+(v_2) = 4n + 4m - 3$
$f^+(v_i) = i + 2n + 3m - 4, \qquad 3 \le i \le n-1$
$f^+(v_n) = 4m + 4n - 5$;
$f^+(v_1') = m + n - 1$
$f^+(v_i') = 2n + m - i - 1, \qquad 2 \le i \le n-1$
$f^+(v_n') = 3(m+n) - 3$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^+ : V(P_m^+ \cup P_n^+) \to \{0,1,2,\dots,2p-1\}$. Hence, f is a strong edge-graceful labeling.

Thus, the graph $P_m^+ \cup P_n^+$ is a strong edge-graceful graph for all $m \ge 4, n \ge 3$.

Illustration 2.8

Strong edge-graceful labeling of $P_6^+ \cup P_6^+$

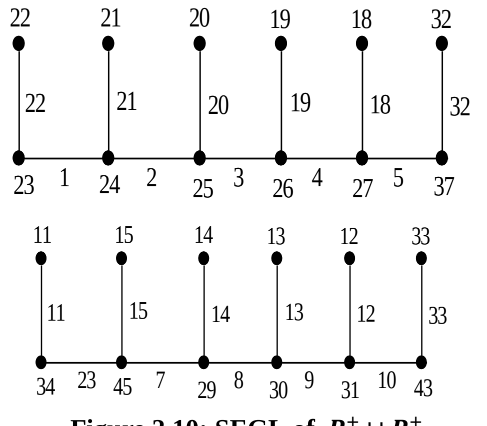

Figure 2.10: SEGL of $P_6^+ \cup P_6^+$

Theorem 2.9

The graph $P_m^+ \cup C_n^+$, $(n, m \ge 3)$ is a strong edge-graceful graph.

Proof

Let $\{u_i, u_i', v_j, v_j' \mid 1 \le i \le m, 1 \le j \le n\}$ be the vertices and $\{e_i, e_j' \mid 1 \le i \le 2m-1, 1 \le j \le 2n\}$ be the edges of $P_m^+ \cup C_n^+$ as shown in Figure 2.11.

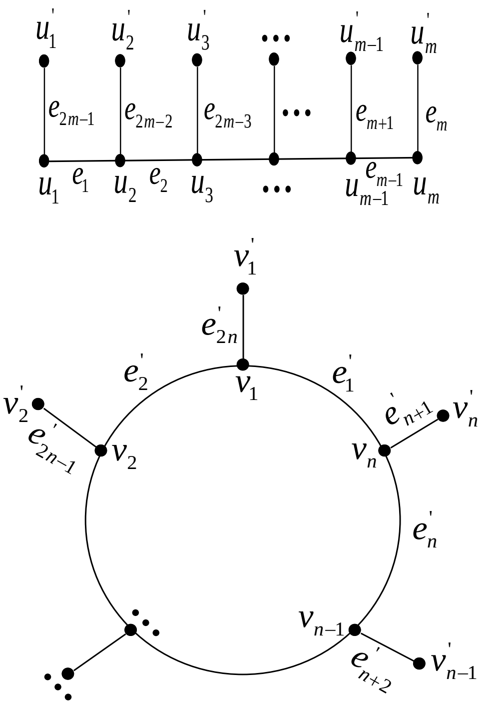

Figure 2.11: Ordinary labeling of $P_m^+ \cup C_n^+$

We first label the edges as follows:

Define $f: E(P_m^+ \cup C_n^+) \to \left\{1,2,\dots,\left[\frac{3q}{2}\right]\right\}$ by

$f(e_i) = i, \qquad 1 \le i \le m-1$

$f(e_m) = 3(m+n) - 4$

$f(e_i) = 2n + i, \qquad m+1 \le i \le 2m-1$

$f(e_1') = m$;

$f(e_2') = 2n + 2m$

$f(e_i') = i + m - 2, \quad 3 \le i \le 2n-2,\ i \ne n+1$

$f(e_{n+1}') = 3(m+n) - 3$

$f(e_{2n-1}') = \begin{cases} 2n+1, & m = 3 \\ 2n+m-3, & m \ne 3 \end{cases}$

$f(e_{2n}') = m + n - 1$

Then the induced vertex labels are:

$f^+(u_i) = 2n + 2m + i - 1, \qquad 1 \le i \le m-1$

$f^+(u_m) = 4m + 3n - 5$

$f^+(u_i') = 2n + 2m - i, \qquad 1 \le i \le m-1$

$f^+(u_m') = 3m + 3n - 4$;

$f^+(v_1) = 4m + 3n - 1$

$f^+(v_2) = \begin{cases} 4n+4m-1, & m = 3 \\ 4n+4m-2, & m \ne 3 \end{cases}$

$f^+(v_i) = 2n + 3m + i - 4, \qquad 3 \le i \le n-1$

$f^+(v_n) = \begin{cases} 4n+5m-5 & m = 3,4 \\ m-5 & m \ge 5 \end{cases}$

$f^+(v_1') = m + n - 1$

$f^+(v_2') = \begin{cases} 2n+m-2, & m = 3 \\ 2n+m-2 & m \ne 3 \end{cases}$

$f^+(v_i') = 2n + 3m - i - 1, \qquad 3 \le i \le n-1$

$f^+(v_n') = 3m + 3n - 3,$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^+: V(P_m^+ \cup C_n^+) \to \{0,1,2,\dots,2p-1\}$. Hence, f is a strong edge-graceful labeling.

Thus, the graph $P_m^+ \cup C_n^+$ is a strong edge-graceful graph for all $m, n \ge 3$

Illustration 2.10

Strong edge-graceful labeling of $P_5^+ \cup C_7^+$

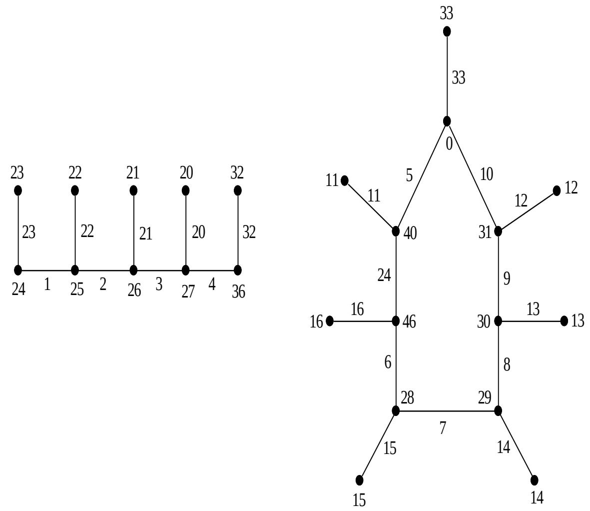

Figure 2.12: SEGL of $P_5^+ \cup C_7^+$

Theorem 2.11

The graph $(K_2 \odot C_n) \cup (K_2 \odot C_n), (n \ge 3)$ is a strong edge-graceful graph.

Proof

Let $\{e_i, e_i' | 1 \le i \le 2n+1\}$ and $\{u_i, v_i | 1 \le i \le 2n\}$ be the edges and the vertices of $(K_2 \odot C_n) \cup (K_2 \odot C_n)$ as shown in Figure 2.13.

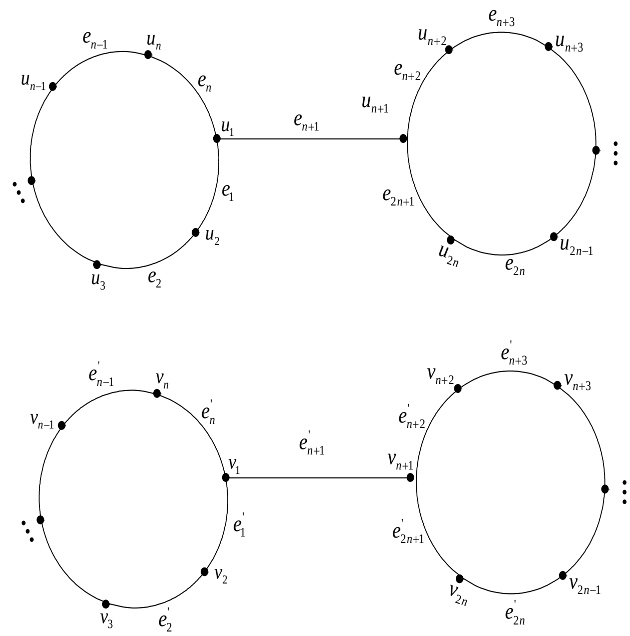

Figure 2.13: Ordinary labeling of $K_2 \odot C_n \cup K_2 \odot C_n$

We first label the edges as follows:
Define $f: E((K_2 \odot C_n) \cup (K_2 \odot C_n)) \to \left\{1,2,\dots,\left\lceil\frac{3q}{2}\right\rceil\right\}$ by

$f(e_i) = i,\qquad 1 \le i \le 2n+1$
$f(e_i') = 2i + 2n,\qquad 1 \le i \le n$
$f(e_{n+1}') = 4n+2$
$f(e_i') = 2i-1,\qquad n+2 \le i \le 2n+1$

Then the induced vertex labels are:

$f^+(u_1) = 2n+2$;
$f^+(u_i) = 2i-1,\qquad 2 \le i \le n$
$f^+(u_{n+1}) = 4n+4$
$f^+(u_i) = 2i+1,\qquad n+2 \le i \le 2n$
$f^+(v_1) = 2n+4$
$f^+(v_i) = 4i+4n-2,\qquad 2 \le i \le n$
$f^+(v_{n+1}) = 2n+6$
$f^+(v_i) = 4i,\qquad n+2 \le i \le 2n-1$
$f^+(v_{2n}) = 0$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^+: V((K_2 \odot C_n) \cup (K_2 \odot C_n)) \to \{0,1,2,\dots,2p-1\}$. Hence, f is a strong edge-graceful labeling.

Thus, the graph $(K_2 \odot C_n) \cup (K_2 \odot C_n)$ is a strong edge-graceful graph for all $n \ge 3$.

Illustration 2.12

SEGL of $(K_2 \odot C_4) \cup (K_2 \odot C_4)$

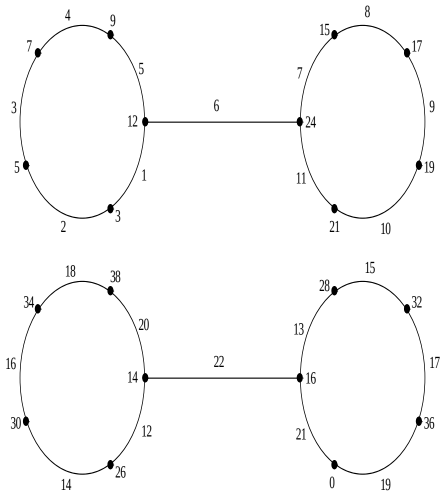

Figure 2.14: SEGL of $K_2 \odot C_5 \cup K_2 \odot C_5$

Theorem 2.13

The graph $F_n \cup F_n$, $(n \ge 3)$ is a strong edge-graceful graph.

Proof

Let $\{u, v, u_i, v_i | 1 \le i \le n\}$ and $\{e_i, e_i' | 1 \le i \le 2n-1\}$ be the vertices and the edges of $F_n \cup F_n$ as shown in Figure 2.15. We note that $|V(F_n \cup F_n)| = 2n+2$ and $|E(F_n \cup F_n)| = 4n-2$.

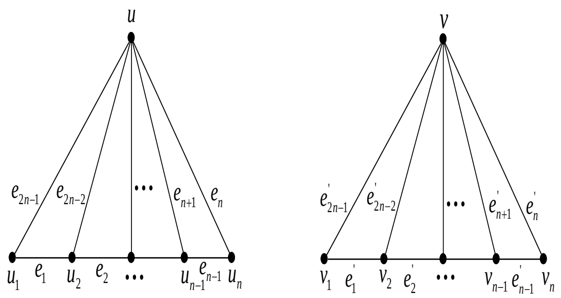

Figure 2.15: Ordinary labeling of $F_n \cup F_n$

Case 1: n is odd $(n \ge 7)$

We first label the edges as follows:
Define $f: E(F_n \cup F_n) \to \{1,2,\dots,\left\lceil\frac{3q}{2}\right\rceil\}$ by

For $1 \le i \le n-1$

$$f(e_i) = \begin{cases} \dfrac{i+1}{2}, & i\ odd \\ \dfrac{8n-i+8}{2}, & i\ even \end{cases}$$

$$f(e_n) = \begin{cases} 4n+4, & if\ n \ne 11 \\ 4n+5, & if\ n = 11 \end{cases}$$

For $n+1 \le i \le 2n-1$

$$f(e_i) = \begin{cases} \dfrac{6n+11-i}{2}, & i\ odd \\ \dfrac{i+2n-2}{2}, & i\ even \end{cases}$$

For $1 \le i \le n-1$

$$f(e_i') = \begin{cases} \dfrac{n+i}{2}, & i\ odd \\ \dfrac{7n-i+9}{2}, & i\ even \end{cases}$$

$$f(e_n') = 4n+9$$

For $n+1 \le i \le 2n-1$

$$f(e_i') = \begin{cases} \dfrac{7n+10-i}{2}, & i\ odd \\ \dfrac{n+i-1}{2}, & i\ even \end{cases}$$

Then the induced vertex labels are:

$$f^+(u) = \begin{cases} 1, & if\ \ n = 11 \\ 0, & if\ \ n \ne 11 \end{cases}$$

For $1 \le i \le n-1$

$$f^+(u_i) = \begin{cases} \dfrac{4n+i+13}{2}, & i\ odd \\ \dfrac{4n-i-2}{2}, & i\ even \end{cases}$$

$$f^+(u_n) = \begin{cases} \dfrac{7n+9}{2}, & if\ n \ne 11 \\ \dfrac{7n+11}{2}, & if\ n = 11 \end{cases}$$

$$f^+(v) = 5$$
$$f^+(v_1) = 3n+6$$

For $2 \le i \le n-1$

$$f^+(v_i) = \begin{cases} \dfrac{5n+12+i}{2}, & i\ odd \\ \dfrac{3n-i-1}{2}, & i\ even \end{cases}$$

$$f^+(v_n) = 3n + 10$$

Case 2: $\boldsymbol{n}$ is even $(\boldsymbol{n \geq 6})$

For $1 \leq i \leq n-1$

$$f(e_i) = \begin{cases} \dfrac{i+1}{2}, & i\ odd \\ \dfrac{8n-i+8}{2}, & i\ even \end{cases}$$

For $n \leq i \leq 2n-1$ and $i \neq 2n-2$

$$f(e_i) = \begin{cases} \dfrac{6n-i+9}{2}, & i\ odd \\ \dfrac{2n+i}{2}, & i\ even \end{cases}$$

$$f(e_{2n-2}) = 2n+2$$

For $1 \leq i \leq 2n-1$

$$f(e_i') = \begin{cases} \dfrac{7n-i+9}{2}, & i\ odd \\ \dfrac{n+i}{2}, & i\ even \end{cases}$$

Then the induced vertex labels are:

$$f^+(u) = 3;$$

$f^+(u_1) = 2n+6$; $f^+(u_2) = 2n+2$

For $3 \leq i \leq n-1$

$$f^+(u_i) = \begin{cases} \dfrac{4n+i+11}{2}, & i\ odd \\ \dfrac{4n-i}{2}, & i\ even \end{cases}$$

$f^+(u_n) = 2n$; $\qquad f^+(v) = 0$

$f^+(v_1) = 2n+5$;

$f^+(v_2) = \frac{3n}{2}$

For $3 \leq i \leq n-1$

$$f^+(v_i) = \begin{cases} \dfrac{5n+i+9}{2}, & i\ odd \\ \dfrac{3n-i+2}{2}, & i\ even \end{cases}$$

$$f^+(v_n) = 1$$

Case 3: $\boldsymbol{n = 3, 4, 5}$

Strong edge-graceful labeling of $F_3 \cup F_3$, $F_4 \cup F_4$ and $F_5 \cup F_5$ are shown in [Figures 7.16-Figure 2.18] respectively.

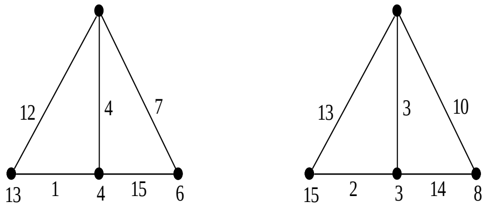

Figure 2.16: SEGL of $\boldsymbol{F_3 \cup F_3}$

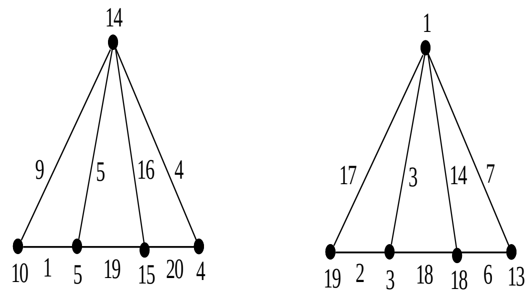

Figure 2.17: SEGL of $\boldsymbol{F_4 \cup F_4}$

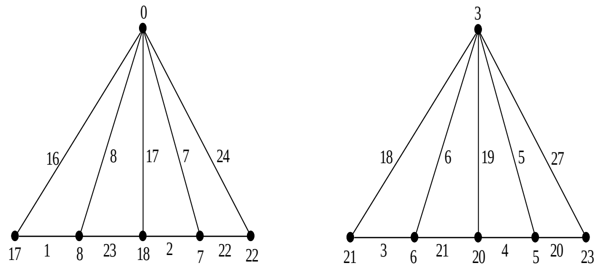

Figure 2.88: SEGL of $\boldsymbol{F_5 \cup F_5}$

Clearly, all vertex labels are distinct. Hence, the above defined edge labeling function induces the vertex labeling function $f^+: V(F_n \cup F_n) \to \{0,1,2,\dots,2p-1\}$. Hence, f is a strong edge-graceful labeling.

Thus, the graph $F_n \cup F_n$ is a strong edge-graceful graph for all $n \geq 3$.

Illustration 2.14

Strong edge-graceful labeling of $F_6 \cup F_6$ and $F_7 \cup F_7$ are shown in Figure 2.29.

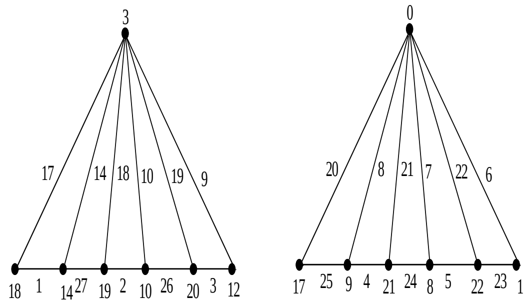

Figure 2.19: SEGL of $\boldsymbol{F_6 \cup F_6}$

Theorem 2.15

The graph $K_{1,m} \cup K_{1,n}$ of odd order $p \geq 9$ is a strong edge-graceful graph.

Proof

Let $\{v_0, v_i, v_0', v_j' | 1 \leq i \leq m, 1 \leq j \leq n\}$ be the vertices and $\{e_i, e_j' | 1 \leq i \leq m,\ 1 \leq j \leq n\}$ be the edges of $K_{1,m} \cup K_{1,n}$ as shown in Figure 2.20. We note that $P = |V(K_{1,m} \cup K_{1,n})| = m+n+2$ and $V = |E(K_{1,m} \cup K_{1,n})| = m+n$.

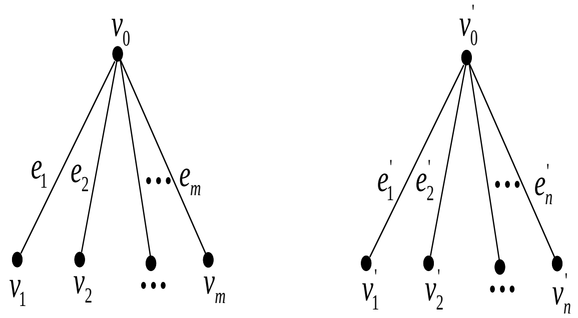

Figure 2.20: Ordinary labeling of $K_{1,m} \cup K_{1,n}$

Case 1: $p > 9$

The graph $K_{1,m} \cup K_{1,n}$ is of odd order only if either m is even end n is odd or vice-versa. With loss of generality, let m be odd and n be even.

Now consider the Diophantine equation $x_1 + x_2 = 2p$ and the solution of the equations are of the form $(t, 2p - t)$ where $\frac{p+7}{2} \leq t \leq p - 1$, the number of pairs of solutions are $\frac{p-7}{2}$.

With these pair of solutions, label the edges $\{e_i : 4 \leq i \leq m\}$ of $K_{1,m}$ and the edges $\{e_i' : 3 \leq i \leq n\}$ of $K_{1,n}$ by the coordinates of the pairs in any order so that adjacent edges receive the coordinates of the pair.

Now we label the remaining edges as follows:

$f(e_1) = 1$; $f(e_2) = 2$;
$f(e_3) = 5$
$f(e_1') = 4$; $f(e_2') = 3$

Then the induced vertex labels are:

$f^+(v_0) = 8$; $f^+(v_0') = 7$

and all the pendant vertices will receive labels of the edges with which they are incident and they are distinct.

Case 2: $n = 4, m = 3$

Strong edge-graceful labeling of $K_{1,3} \cup K_{1,4}$ is shown in Figure 2.21.

Clearly, all vertex labels are distinct.

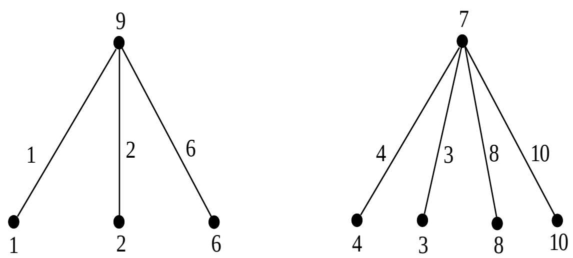

Figure 2.21: SEGL of $K_{1,3} \cup K_{1,4}$

Hence, $K_{1,m} \cup K_{1,n}$ is a strong edge-graceful graph for $p \geq 9$.

Illustration 2.16

Strong edge-graceful labeling of $K_{1,5} \cup K_{1,6}$ and $K_{1,3} \cup K_{1,8}$ are shown in Figure 2.33.

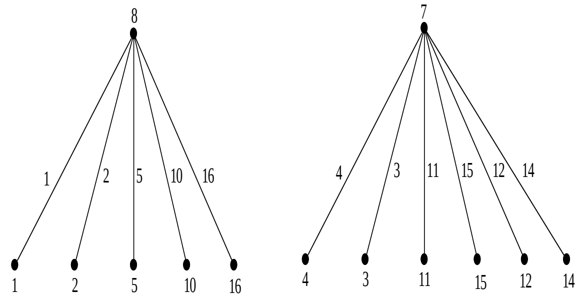

Figure 2.22: SEGL of $K_{1,5} \cup K_{1,6}$

Theorem 2.17

The graph $B_{m,n} \cup B_{m,n}$ is a strong edge-graceful graph for all $m, n \geq 3$.

Proof

Let $\{u, v, u', v', u_i, v_j, u_i', v_j' \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$ and $\{e_i, e_j', a_i, a_j', a, e \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$ be the vertices and the edges of $B_{m,n} \cup B_{m,n}$ as shown in Figure 2.23.

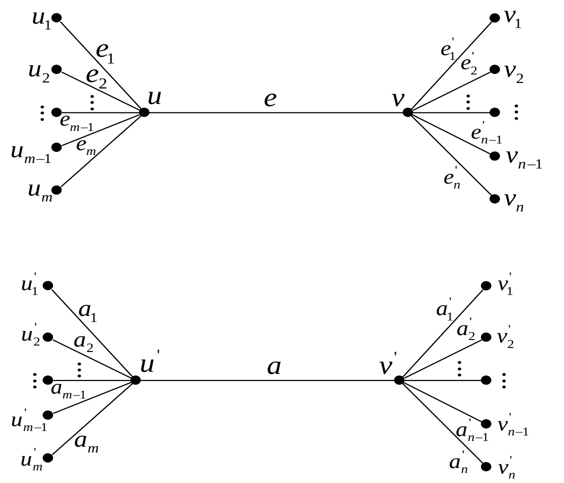

Figure 2.23: Ordinary labeling of $B_{m,n} \cup B_{m,n}$

Now consider the Diophantine equation $x_1 + x_2 = 2p$ and the solution of the equation are of the form $(t, 2p - t)$ where $\frac{p+6}{2} \leq t \leq p - 1$, the number of pairs of solutions are $\frac{p-6}{2}$.

Case 1: m is odd and n is even (or) n is odd, m is even

Without loss of generality, assume m is even or n is odd. With $\frac{p-6}{2}$ pairs of solution, we label the edges $\{e_i : 4 \leq i \leq m\}$, $\{e_i' : 2 \leq i \leq n\}$ and $\{a_i, a_j' \,|\, 1 \leq i \leq m, 2 \leq j \leq n\}$ by the coordinates of the pairs in any order so that adjacent edges receive the coordinates of the pair.

Now we label the remaining edges as follows:

$f(e_1) = 1$; $f(e_1') = 5$; $f(a) = 4$; $f(a_1') = 3$

Then the induced vertex labels are:
$f^+(u) = 1$; $f^+(v) = 6$; $f^+(u') = 4$;
$f^+(v') = 7$;
and all the pendant vertices will receive labels of the edges with which they are incident and they are distinct.

Case 2: Both m and n are odd

With $\frac{p-6}{2}$ pairs of solutions, we label the edges $\{e_i, e_j', a_i, a_j' \mid 2 \le i \le m, 2 \le i \le n\}$ by the coordinates of the pairs in any order so that adjacent edges receive the coordinates of the pair.
Now, we label the remaining edges as follows:
$f(e_1) = 9$; $f(e) = 1$; $f(e_1') = 2$;
$f(a_1) = 5$; $f(a) = 3$; $f(a_1') = 4$
Then the induced vertex labels are:
$f^+(u) = 10$; $f^+(v) = 3$; $f^+(u') = 8$;
$f^+(v') = 7$
and all the pendant vertices will receive labels of the edges with which they are incident and they are distinct.

Case 3: m and n both are even and $p \ge 16$

In $\frac{p-6}{2}$ pairs of solutions, exclude the pair $(\frac{p+6}{2}, \frac{3p-6}{2})$ and we label the edges $\{e_i, a_i, e_j', a_j' \mid 1 \le i \le m, 3 \le j \le n\}$ by the coordinates of pairs in any order so that the adjacent edges receive the coordinates of the pair.
Now, we label the remaining edges as follows:
$f(e) = 1$; $f(e_1') = 4$; $f(e_2') = 5$;
$f(a) = 2$; $f(a_1') = 3$; $f(a_2') = 6$

Then the induced vertex labels are:
$f^+(u) = 1$; $f^+(v) = 10$; $f^+(u') = 2$;
$f^+(v') = 11$
and all the pendant vertices will receive labels of the edges with which they are incident and they are distinct.

Case 4: m and n both even and $p < 16$

The only possibility of this case is $B_{2,2} \cup B_{2,2}$ and the strong edge-graceful labeling of $B_{2,2} \cup B_{2,2}$ is shown in Figure 2.24.

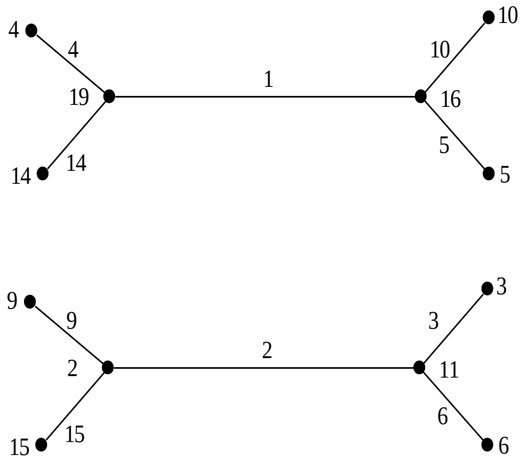

Figure 2.24: SEGL of $B_{2,2} \cup B_{2,2}$

Illustration 2.18

Strong edge-graceful labeling $B_{5,5} \cup B_{5,5}$

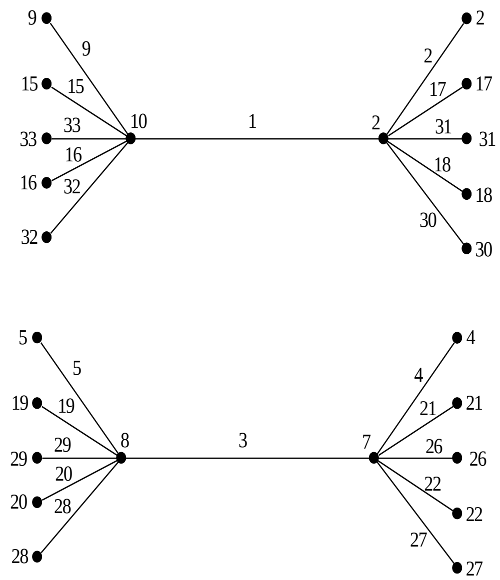

Figure 2.25: SEGL of $B_{5,5} \cup B_{5,5}$

3. CONCLUSION

In this paper two same or different family of disconnection graphs has been discussed. Further this leads to the open study about how, more than two disconnected graphs behave.

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